Quotient Types by Idempotent Functions in Cedille
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Guide
What is a Type Theory?
- Foundational theory for logic, math and computer science
- Commonly built on the lambda calculus
- Most common core is the Calculus of Constructions (CoC) (with function types, type quantification, etc)

Traditional Type Theory
(e.g. Agda and Coq)
CoC
Dependent Pair
\[ \sum \]
Inductive Identity Type
\[ Id_A \ a \ b \]
Inductive Definitions

Cedille
CoC
\[ \forall \]
Implicit (or erased) Function Types
\[ \sim \]
Primitive Equality
\[ \Pi \]
Dependent Intersections

What is a Definable Quotient Type?
Given A and \( \sim \)
Define: \( f : A \to A \) such that
\[ \forall a, b \in A. \text{if } a \sim b \text{ then } f(a) = f(b) \]
\[ \forall a \in A. f(a) \sim a \]
Then \( Q = \sum a \in A. Id_A(f(a) \ a) \)
Example:
\[ (N, N) \to (N, N) \]
\[ f = \text{reduce to irreducible fraction} \]
\[ Q = \sum (n, d) \in (N, N). Id_{(N, N)} f((n, d)) \ (n, d) \]

Discussion of Definable Quotient Types
- Not all quotients are definable in the above sense (e.g. unordered pairs)
- Not all definable quotients can be defined as inductive types
- Quotients give a different perspective on constructing types

Quotients give two different meanings for a concept:
- Setoid: \( (N, N) \) and \( \sim \)
- Set: \( Q \) and \( Id_Q \)
Proofs can benefit from both
Projections can get in the way of equalities between views:
\[ f((n, m)) = (n, m) \]
\[ (n, m) = \pi_1 q \]
where \( \pi_1 \) is the first projection

All the same benefits and \( Q \) is a subtype of \( A \)
\[ \forall q \in Q. \exists a \in A. q = a \]
This prevents projections from interfering with equalities between \( Q \) and \( A \)
\[ f((n, m)) = (n, m) \]
\[ (n, m) = q \]

Conclusions
1. Quotients by idempotent functions hide the equivalence relation \( \sim \)
2. Cedille’s encoding of quotients by idempotent functions gives you, additionally, subtypes

Related Work
1. Nuo Li, Quotient Types in Type Theory
2. Cyril Cohen, Pragmatic Quotient Types in Coq
3. Homotopy Type Theory for an alternative approach to quotients in a nontraditional theory